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Runge's Theorem in Hypercomplex Function Theory

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1. INTRODUCTION

In the last decades important problems have been solved concerning rational approximation of holomorphic functions on a compact subset of the plane. One way of approaching these problems has been by means of functional analysis methods, emerging from the study of certain Banach algebras of functions (see, e.g., [9–11]).

In this paper similar methods are used to prove approximation theorems for the solutions of a hypercomplex differential equation in the $(m+1)$-dimensional Euclidean space. The operator D^k considered is a generalized Cauchy–Riemann operator acting on functions f defined in an open non-empty subset Ω of $\mathbb{R}^{m+1}$ and with values in a Clifford algebra $\mathscr{A}$ constructed over a quadratic n-dimensional real vector space ($m \leqslant n$). It should be noted that the operator D^k may be identified with a strongly elliptic system of homogeneous differential operators (Section 2). Note also that the solutions of the equation $D^k f = 0$ form a subclass of the set of vector-valued poly-harmonic functions and that, when $m = n = 1$, the class of polyanalytic functions of a complex variable is thus obtained.

We first prove that if K is a compact subset of $\mathbb{R}^{m+1}$, then any function which satisfies $D^k f = 0$ in an open neighborhood of K can be uniformly approximated on K by a sequence of polynomial solutions of the equation (if co K is connected) or by a sequence of "rational" solutions having their singularities off K (K arbitrary). Using these results, we generalize the well-known Runge and Hartorgs–Rosenthal theorems from holomorphic function theory (Section 3). Finally, in Section 4 we prove a Runge type theorem for regular solutions at infinity of the operator D^k.

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In this context we also wish to draw attention to the analogues of Runge's theorem established by Chauveheid [3] and du Plessis [7] for harmonic functions and more generally by Chauveheid [2] for the solutions of elliptic differential operators with constant coefficients.

2. Preliminaries

Let $\mathscr{A}$ be the Clifford algebra constructed over a quadratic n-dimensional real vector space with orthogonal basis $\{e_1, ..., e_n\}$. Let furthermore $e_A = e_{i_1} e_{i_2} \cdots e_{i_h}$ be an arbitrary basic element of $\mathscr{A}$ where $A \in \mathscr{P}N$, $N = \{1, 2, ..., n\}$ and $i_1 < i_2 < \cdots < i_h$. Then in [5] an inner product $(\ ,\ )_0$, a norm $|\cdot|_0$ and an involution have been defined on $\mathscr{A}$ turning it into a finite-dimensional H^*-algebra. Recall that for each $\lambda = \sum_A \lambda_A e_A \in \mathscr{A}$, $|\lambda|_0^2 = 2^n \sum_A \lambda_A^2$. Let $m \leqslant n$, $m \neq 0$ and Ω be an open nonempty subset of $\mathbb{R}^{m+1}$. Then in [5] we have established some properties of the space $M_k(\Omega; \mathscr{A})$, consisting of those functions $f \in C_k(\Omega; \mathscr{A})$ for which $D^k f = 0$ in Ω, where $k \in \mathbb{N}$ and $D = \sum_{i=0}^{m} e_i(\partial/\partial x_i)$. Among other things, it has been shown that $M_k(\Omega; \mathscr{A})$, provided with the topology of uniform compact convergence, is a right $\mathscr{A}$-Fréchet module.

Note that, as $\dim \mathscr{A} = 2^n$, the equation $D^k f = 0$ is equivalent to a system of 2^n linear partial differential equations, each of order k, in the 2^n unknown real valued functions f_A.

If the basic elements e_A of $\mathscr{A}$ are ordered in a certain way, then by means of the left regular representation of $\mathscr{A}$, we may associate to each $\lambda \in \mathscr{A}$ a $2^n \times 2^n$ real matrix $\Theta(\lambda)$. Remark that, as $\mathscr{A}$ has an identity e_0, this representation is an isomorphism.

Using the fact that $D\bar{D} = \bar{D}D = \Delta e_0$ where $\bar{D} = \sum_{i=0}^{m} \bar{e}_i(\partial/\partial x_i) = e_0(\partial/\partial x_0) - \sum_{j=1}^{m} e_j(\partial/\partial x_j)$ and $\Delta = \sum_{i=0}^{m} (\partial^2/\partial x_i^2)$, we have

Proposition 2.1. *The system of differential equations associated to the hypercomplex differential operator D^k is strongly elliptic.*

Proof. Since $\Theta(D^k) = (\Theta(D))^k$, it suffices to prove that $\Theta(D)$ is a strongly elliptic system (see, e.g., [8]). Also note that the order of $\Theta(D)$ equals one.

We now claim that for each $j = 1, ..., n$, $\Theta(e_j)$ is a skew-symmetric matrix. Indeed, taking account of the multiplication rules in $\mathscr{A}$ (see [5]), one may easily check that for each $A, B \in \mathscr{P}N$, the element $a_{B,A}$ at the Bth row and Ath column of $\Theta(e_j)$ is equal to $-a_{A,B}$. Hence, if $\Theta(e_j)^T$ denotes the transpose of $\Theta(e_j)$, $\Theta(e_j)^T + \Theta(e_j) = 0$ or $\Theta(e_j)^T = \Theta(-e_j) = \Theta(\bar{e}_j)$, $j = 1, ..., n$. As $\Theta(e_0) = \delta$, the identity matrix, we so obtain that $\Theta(D)^T = \Theta(\bar{D})$. Hence $\Theta(\bar{D}D) = \Theta(\bar{D})\, \Theta(D) = \Delta\delta$ so that $(\det(\Theta(D))^2 = \Delta^{2n}$. From this relation it clearly follows that $\Theta(D)$ is a strongly elliptic system. ∎

In view of the foregoing proposition, we can freely use basic results from the general theory of strongly elliptic systems of differential operators. In this connection we recall that in [6], a bounded left (right) $\mathscr{A}$-linear functional $T^{(l)}(T^{(r)})$ on the bi-$\mathscr{A}$-module $\mathscr{D}_\infty(\Omega, \mathscr{A})$ was said to be a left (right) $\mathscr{A}$-distribution in Ω. For further details concerning this type of matrix distributions, we refer the reader to [6].

In the next sections, a main role will be played by the (right) Cauchy transform of an $\mathscr{A}$-valued measure in $\mathbb{R}^{m+1}$ with compact support. For the sake of completeness, we here recall its definition.

Let E_k be the fundamental solution of D^k (see, e.g., [6]) and μ be an $\mathscr{A}$-valued measure in $\mathbb{R}^{m+1}$ with compact support. Then $\mu * E_k$ with

$$\mu * E_k(x) = \int d\mu(t)\, E_k(x - t)$$

is the right Cauchy transform of μ.

Note that $\mu * E_k \in L_1^{\text{loc}}(\mathbb{R}^{m+1}; \mathscr{A})$ and that, considered as a right $\mathscr{A}$-distribution in $\mathbb{R}^{m+1}$, $(\mu * E_k)\, D^k = 0$ in $\text{co}[\mu]$. Moreover, as the system of differential operators associated to D^k is strongly elliptic, $\mu * E_k \in C_\infty(\text{co}[\mu]; \mathscr{A})$ and $(\mu * E_k)\, D^k = 0$ in $\text{co}[\mu]$, which implies that $\mu * E_k$ is an analytic $\mathscr{A}$-valued function in $\text{co}[\mu]$.

3. Runge's Theorem

We first establish some results concerning "rational" approximation of functions f which satisfy $D^k f = 0$ in a neighborhood of a compact set $K \subset \mathbb{R}^{m+1}$.

Definition 3.1. Let K be a compact subset of $\mathbb{R}^{m+1}$. Then we call $M_k(K; \mathscr{A})$ the set of functions f for which there exists an open neighborhood ω of K such that $f \in M_k(\omega; \mathscr{A})$.

It is clear from the definition that $M_k(K; \mathscr{A})$ is a right $\mathscr{A}$-module.

The following lemma is fundamental for the sequel. It gives necessary and sufficient conditions for an $\mathscr{A}$-valued measure in $\mathbb{R}^{m+1}$ supported on K to be an annihilating measure for $M_k(K; \mathscr{A})$.

Lemma 3.1. *Let μ be an $\mathscr{A}$-valued measure in $\mathbb{R}^{m+1}$ with support contained in the compact subset K of $\mathbb{R}^{m+1}$. Then $\int d\mu(x) f(x) = 0$ for all $f \in M_k(K; \mathscr{A})$ iff $\mu * E_k = 0$ in* co K.

Proof. As to the necessary condition, let a be an arbitrary point in co K and put $f(x) = E_k(a - x)$. Then clearly $f \in M_k(K; \mathscr{A})$ and $\mu * E_k(a) = \int d\mu(x)\, E_k(a - x) = 0$. Hence $\mu * E_k = 0$ in co K. Conversely, let

$f \in M_k(K; \mathscr{A})$. Then there exists an open neighborhood ω of K such that $f \in M_k(\omega; \mathscr{A})$. Take $\phi \in \mathscr{D}_\infty(\omega; \mathbb{R})$ with $\phi(x) = 1$ in ω_1 where ω_1 is open and $K \subset \omega_1 \subset \omega$. Then $f\phi \in \mathscr{D}_\infty(\omega; \mathscr{A})$. If μ is considered as a right $\mathscr{A}$-distribution $\mu^{(r)}$, we have on the one side that

$$\langle \mu^{(r)}, f\phi \rangle = \int d\mu(x) f\phi(x) = \int d\mu(x) f(x),$$

while on the other side (see also [6])

$$\begin{aligned} \langle \mu^{(r)}, f\phi \rangle &= \langle \mu^{(r)} * E_k^{(r)} D^k, f\phi \rangle = \langle (\mu^{(r)} * E_k^{(r)})\, D^k, f\phi \rangle \\ &= (-1)^k \langle \mu^{(r)} * E_k^{(r)}, D^k(f\phi) \rangle = 0. \end{aligned}$$

Hence $\int d\mu(x) f(x) = 0$. ∎

The following results all deal with density properties of certain $\mathscr{A}$-modules in either $M_k(K; \mathscr{A})$ or $M_k(\Omega; \mathscr{A})$. In their proofs, generalizations are used of the Hahn–Banach and Riesz representation theorems which may be found in [4, 6], respectively.

LEMMA 3.2. *Let K be a compact subset of $\mathbb{R}^{m+1}$ whose complement is connected, $f \in M_k(K; \mathscr{A})$ and $\epsilon > 0$. Then there exists $g \in M_k(\mathbb{R}^{m+1}; \mathscr{A})$ such that* $\sup_{x \in K} |f(x) - g(x)|_0 \leqslant \epsilon$, i.e., *$M_k(\mathbb{R}^{m+1}; \mathscr{A})$ is uniformly dense in $M_k(K; \mathscr{A})$.*

Proof. Let $B(0, R_1) = B_1$ be an open ball in $\mathbb{R}^{m+1}$ such that $K \subset B_1$. Then obviously, $M_k(\bar{B}_1; \mathscr{A})$ is a submodule of $M_k(K; \mathscr{A})$. We now claim that $M_k(\bar{B}_1; \mathscr{A})$ is uniformly dense in $M_k(K; \mathscr{A})$.

By means of the Hahn–Banach and Riesz representation theorems, it clearly suffices to prove that each $\mathscr{A}$-valued measure μ in $\mathbb{R}^{m+1}$ supported on K which annihilates $M_k(\bar{B}_1; \mathscr{A})$ is also zero on $M_k(K; \mathscr{A})$.

Let μ be such a measure. Then by assumption, $\int d\mu(x)\, h(x) = 0$ for all $h \in M_k(\bar{B}_1; \mathscr{A})$ so that in view of Lemma 3.1 $\mu * E_k = 0$ in co $\bar{B}_1$. Since $\mu * E_k$ is analytic in co K and co K is connected, we thus have that $\mu * E_k = 0$ in co K so that, once more by Lemma 3.1, $\int d\mu(x) f(x) = 0$ for all $f \in M_k(K; \mathscr{A})$ or μ annihilates $M_k(K; \mathscr{A})$.

As a second step, we prove that $M_k(\mathbb{R}^{m+1}; \mathscr{A})$ is uniformly dense in $M_k(K; \mathscr{A})$. Consider therefore a sequence $(\bar{B}_i)_{i=1}^\infty$ of closed balls with $K \subset \mathring{B}_1 \subset \bar{B}_1 \subset \mathring{B}_2 \subset \bar{B}_2 \subset \cdots$ such that $\mathbb{R}^{m+1} = \bigcup_{i=1}^\infty \mathring{B}_i = \bigcup_{i=1}^\infty \bar{B}_i$, and put for convenience $K = \bar{B}_0$.

In an analogous way as before, for each $i \in \mathbb{N}$, $M_k(\bar{B}_{i+1}; \mathscr{A})$ is uniformly dense in $M_k(\bar{B}_i; \mathscr{A})$.

Let now $f = f_0 \in M_k(K; \mathscr{A})$. By a successive application of the first step in the proof, a sequence $(f_i)_{i \in \mathbb{N}}$ may be found such that for each $i \in \mathbb{N}$,

$f_i \in M_k(\bar{B}_i ; \mathscr{A})$ and $\sup_{x\in\bar{B}_i} |f_i(x) - f_{i+1}(x)|_0 \leqslant \epsilon 2^{-i-1}$. Take now $i \in \mathbb{N}$ fixed and consider in $M_k(\bar{B}_i ; \mathscr{A})$ the sequence $(f_{i+k})_{k=0}^{\infty}$. Then this is a Cauchy sequence in $M_k(\bar{B}_i ; \mathscr{A})$ since for $s < t$,

$$\begin{aligned}
\sup_{x\in\bar{B}_i} |f_s(x) - f_t(x)|_0 &\leqslant \sup_{x\in\bar{B}_s} |f_s(x) - f_t(x)|_0 \\
&\leqslant \sup_{x\in\bar{B}_s} |f_s(x) - f_{s+1}(x) + f_{s+1}(x) - \cdots + f_{t-1}(x) - f_t(x)|_0 \\
&\leqslant \sup_{x\in\bar{B}_s} |f_s(x) - f_{s+1}(x)|_0 + \cdots + \sup_{x\in\bar{B}_{t-1}} |f_{t-1}(x) - f_t(x)|_0 \\
&\leqslant \epsilon \sum_{j=s+1}^{t} \frac{1}{2^j}.
\end{aligned}$$

It hence remains a Cauchy sequence in $M_k(\mathring{B}_i ; \mathscr{A})$ so that, as $M_k(\mathring{B}_i ; \mathscr{A})$ is a right Fréchet $\mathscr{A}$-module, there exists $g_i \in M_k(\mathring{B}_i , \mathscr{A})$ such that $f_{i+k} \to g_i$ in $M_k(\mathring{B}_i ; \mathscr{A})$. Moreover, as the sequence $(f_{i+k})_{k=0}^{\infty}$ is a subsequence of $(f_{i-1+k})_{k=0}^{\infty}$, the latter being convergent in $M_k(\mathring{B}_{i-1} ; \mathscr{A})$ to some g_{i-1}, we have that g_i restricted to $\mathring{B}_{i-1}$ coincides with g_{i-1}. So we have obtained a sequence $(g_i)_{i=1}^{\infty}$ with $g_i \in M_k(\mathring{B}_i ; \mathscr{A})$ for all $i = 1, 2, \ldots$ and $g_i \mid \mathring{B}_{i-1} = g_{i-1}$. As $\mathbb{R}^{m+1} = \bigcup_{i=1}^{\infty} \mathring{B}_i$, for each $x \in \mathbb{R}^{m+1}$ there exists a least index $i \geqslant 1$ such that $x \in \mathring{B}_i$. We may so define a function g in $\mathbb{R}^{m+1}$ by putting $g(x) = g_i(x)$, $x \in \mathbb{R}^{m+1}$. Obviously, $g \in M_k(\mathbb{R}^{m+1}; \mathscr{A})$. Moreover, $\sup_{x\in K} |f(x) - g(x)|_0 \leqslant \epsilon$, since for each $x \in K$,

$$\begin{aligned}
|f(x) - g(x)|_0 &= \lim_{j\to\infty} |f(x) - f_j(x)|_0 \\
&= \lim_{j\to\infty} |f(x) - f_1(x) + f_1(x) - \cdots + f_{j-1}(x) - f_j(x)|_0 \\
&\leqslant \lim_{j\to\infty} (|f(x) - f_1(x)|_0 + \cdots + |f_{j-1}(x) - f_j(x)|_0) \\
&\leqslant \lim_{j\to\infty} \epsilon \left(\sum_{r=1}^{j} \frac{1}{2^r} \right) = \epsilon. \quad \blacksquare
\end{aligned}$$

Remarks. 1. In [5] we have proved that if $f \in M_k(\Omega; \mathscr{A})$ and $a \in \Omega$, then there exists a suitable neighborhood Ω_a of a such that f admits a Taylor development in Ω_a. As compared with the case of holomorphic functions of a complex variable, the role of the powers of $(z - a)$ is taken over by homogeneous polynomials of the form

$$\frac{(x_0 - a_0)^s}{s!} V_{l_1 \cdots l_{p-s}}(x - a), \tag{*}$$

where $p \in \mathbb{N}$, $0 \leqslant s \leqslant \min(p; k - 1)$, $(l_1 ,\ldots, l_{p-s}) \in \{1,\ldots, m\}^{p-s}$ and

$$V_{l_1 \cdots l_{p-s}}(x) = \sum_{\pi(l_1, \ldots, l_{p-s})} z_{l_1} z_{l_2} \cdots z_{l_{p-s}},$$

the sum running over all distinguishable permutations of $l_1, l_2, \ldots, l_{p-s}$ taken all at a time. Hereby, $z_l = x_l e_0 - x_0 e_l$, $l = 1, \ldots, m$. Let us also recall that each of the homogeneous polynomials $(*)$ is in $M_k(\mathbb{R}^{m+1}; \mathscr{A})$.

The Taylor development in a then reads as follows:

$$f(x) = \sum_{p=0}^{\infty} \sum_{s=0}^{\min(p,k-1)} \sum_{(l_1,\ldots,l_{p-s})} \frac{(x_0 - a_0)^s}{s!} V_{l_1 \cdots l_{p-s}}(x - a) \frac{\partial^{p-s} D^s f}{\partial x_{l_1} \cdots \partial x_{l_{p-s}}} \bigg|_{x=a} .$$

It can also be proved that if $f \in M_k(\mathbb{R}^{m+1}; \mathscr{A})$, then its Taylor development in O holds in the whole of $\mathbb{R}^{m+1}$.

2. Let $\mathscr{Q} = \{(x_0^s/s!)\, V_{l_1 \cdots l_{p-s}}(x) \colon p \in \mathbb{N},\quad 0 \leqslant s \leqslant \min(p, k-1),$ $(l_1, \ldots, l_{p-s}) \in \{1, \ldots, m\}^{p-s}\}$ and put $\mathscr{P} = \mathrm{sp}_{\mathscr{A}} \mathscr{Q}$, the right $\mathscr{A}$-span of $\mathscr{Q}$.

In view of the Taylor expansion, Lemma 3.2 may be restated in the the following way: "Let K be a compact subset of $\mathbb{R}^{m+1}$ with co K connected. Then $\mathscr{P}$ is uniformly dense in $M_k(K; \mathscr{A})$."

3. Let G be a suitable open annular domain centred at the origin with associated Laurent domain G^* and suppose that $f \in M_k(G; \mathscr{A})$. Then in analogy with the results from [1], f admits the following Laurent expansion in G^*:

$$f(x) = \sum_{p=0}^{\infty} \sum_{s=0}^{\min(p,k-1)} \sum_{(l_1,\ldots,l_{p-s})} \frac{x_0^s}{s!} V_{l_1 \cdots l_{p-s}}(x)\, \lambda^{(s)}_{l_1 \cdots l_{p-s}}$$

$$- \sum_{p=0}^{\infty} \sum_{s=0}^{\min(p,k-1)} \sum_{(l_1,\ldots,l_{p-s})} \frac{x_0^s}{s!} X_{l_1 \cdots l_{p-s}}(x)\, \mu^{(s)}_{l_1 \cdots l_{p-s}},$$

where again for each $p \in \mathbb{N}$ and $0 \leqslant s \leqslant \min(p, k-1)$, $\lambda^{(s)}_{l_1 \ldots l_{p-s}}, \mu^{(s)}_{l_1 \ldots l_{p-s}} \in \mathscr{A}$. Moreover, for each $(l_1, \ldots, l_{p-s}) \in \{1, \ldots, m\}^{p-s}$,

$$X_{l_1 \cdots l_{p-s}}(x) = \frac{\partial^{p-s}}{\partial u_{l_1} \cdots \partial u_{l_{p-s}}} \frac{\bar{u} - \bar{x}}{\rho^{m+1}} \bigg|_{u=0}$$

with $\rho^2 = \sum_{i=0}^{m} (u_i - x_i)^2$ and $\bar{u} - \bar{x} = \sum_{i=0}^{m} (u_i - x_i)\, \bar{e}_i$.

Note that the functions $(x_0^s/s!)\, X_{l_1 \ldots l_{p-s}}(x)$ all belong to $M_k(\mathbb{R}^{m+1} \backslash \{0\}, \mathscr{A})$; they thus take over the role of the negative powers of z in the case of holomorphic functions of a complex variable. Denote their set by $\mathscr{R}(0)$ and, if $a \in \mathbb{R}^{m+s}$, call $\mathscr{R}(a)$ the set of functions of the form $((x_0 - a_0)^s/s!) \times X_{l_1 \ldots l_{p-s}}(x - a)$, $p \in \mathbb{N}$, $0 \leqslant s \leqslant \min(p, k-1)$, $(l_1, \ldots, l_{p-s}) \in \{1, \ldots, m\}^{p-s}$.

Note also that, since

$$E_k(x) = \frac{1}{\omega_{m+1}} \frac{\bar{x} x_0^{k-1}}{\rho_x^{m+1}(k-1)!}$$

and $D^sE_k = E_{k-s}$ $(0 \leqslant s < k)$ (see [5]), for any $p \in \mathbb{N}$ and $0 \leqslant s \leqslant \min(p, k-1)$,

$$\frac{\partial^{p-s} D^s E_k}{\partial x_{l_1} \cdots \partial x_{l_{p-s}}} = \frac{1}{\omega_{m+1}} \frac{x_0^{k-s-1}}{(k-s-1)!} X_{l_1 \cdots l_{p-s}}.$$

Each of these functions thus has a (k)-pole in 0.

Now let K be an arbitrary compact subset of $\mathbb{R}^{m+1}$ and $\Omega_0, \Omega_1, \Omega_2, \ldots$ be the components of co K, Ω_0 being unbounded. Let furthermore $\mathbf{a} = \{a_i\}$ be a subset of co K which contains one point in each Ω_i, $i = 1, 2, \ldots$ Then we call $\mathscr{R}(\mathbf{a}) = \bigcup_{i \geqslant 1} \mathscr{R}(a_i)$ and $\mathscr{R}^*(\mathbf{a}) = \mathrm{sp}_{\mathscr{A}} \mathscr{R}(\mathbf{a})$.

Clearly the right $\mathscr{A}$-module $M_k(\mathbb{R}^{m+1}; \mathscr{A}) \oplus \mathscr{R}^*(\mathbf{a})$ consists of functions having their singularities ((k)-poles) off K, namely in $\mathbf{a}$.

Using the above notations we have

LEMMA 3.3. *If K is a compact subset of $\mathbb{R}^{m+1}$ and $\mathbf{a}$ is a subset of* co K *having one point in each bounded component of* co K, *then $M_k(\mathbb{R}^{m+1}; \mathscr{A}) \oplus \mathscr{R}^*(\mathbf{a})$ is uniformly dense in $M_k(K; \mathscr{A})$.*

Proof. In view of the Hahn–Banach and Riesz representation theorems, it clearly suffices to prove that if μ is an $\mathscr{A}$-valued measure in $\mathbb{R}^{m+1}$ with compact support contained in K such that $\int d\mu(x)\, h(x) = 0$ for each $h \in M_k(\mathbb{R}^{m+1}; \mathscr{A}) \oplus R^*(\mathbf{a})$, then $\int d\mu(x) f(x) = 0$ for all $f \in M_k(K; \mathscr{A})$.

Let μ be such a measure, consider $\mu * E_K$ and take an arbitrary $i \in \mathbb{N} \backslash \{0\}$. Then, if $a_i \in \Omega_i$, we have that for each $p \in \mathbb{N}$, $0 \leqslant s \leqslant \min(p, k-1)$, $(l_1, \ldots, l_{p-s}) \in \{1, \ldots, m\}^{p-s}$,

$$\begin{aligned}
\left((\mu * E_k) \frac{\partial^{p-s} D^s}{\partial x_{l_1} \cdots \partial x_{l_{p-s}}}\right)(a_i) &= \left(\mu * \frac{\partial^{p-s} D^s E_k}{\partial x_{l_1} \cdots \partial x_{l_{p-s}}}\right)(a_i) \\
&= \int d\mu(x) \frac{\partial^{p-s} D^s E_k}{\partial x_{l_1} \cdots \partial x_{l_{p-s}}} (a_i - x) \\
&= 0.
\end{aligned}$$

As $(\mu * E_k) D^k = 0$ in co$[\mu]$, taking account of the Taylor expansion for $\mu * E_K$ in a neighborhood of a_i and of the fact that Ω_i is connected, we may conclude that $\mu * E_k = 0$ in Ω_i.

Call $K_0 = \mathrm{co}\, \Omega_0$ and take $f_0 \in M_k(K_0; \mathscr{A})$. Then, as K_0 is a compact set whose complement is connected, it follows from Lemma 3.2 that a sequence $(g_j)_{j \in \mathbb{N}}$ in $M_k(\mathbb{R}^{m+1}; \mathscr{A})$ may be found such that $\sup_{x \in K_0} |f_0(x) - g_j(x)|_0 \to 0$ if $j \to \infty$.

Consequently

$$\lim_{j \to \infty} \int d\mu(x) f_0(x) = \lim_{j \to \infty} \int d\mu(x)(f_0(x) - g_j(x)) = 0$$

and so $\int d\mu(x) f_0(x) = 0$ for all $f_0 \in M_k(K_0; \mathscr{A})$, whence $\mu * E_k = 0$ in $\operatorname{co} K_0 = \Omega_0$ (Lemma 3.1.).

We have thus proved that $\mu * E_k = 0$ in $\operatorname{co} K$ so that, again in view of Lemma 3.1, $\int d\mu(x) f(x) = 0$ for all $f \in M_k(K; \mathscr{A})$. ∎

Now let Ω be an open subset of $\mathbb{R}^{m+1}$ and α be a subset of $\operatorname{co} \Omega$ having one point in each component of $\operatorname{co} \Omega$. Furthermore, put $\mathscr{R}(\alpha) = \bigcup_{\mathbf{a} \subset \alpha} \mathscr{R}(\mathbf{a})$, $\mathbf{a}$ being an arbitrary finite or countable subset of α and call $\mathscr{R}^*(\alpha) = \operatorname{sp}_{\mathscr{A}} \mathscr{R}(\alpha)$. Clearly $M_k(\mathbb{R}^{m+1}; \mathscr{A}) \oplus \mathscr{R}^*(\alpha)$ then consists of functions having their singularities ((k)-poles) in $\operatorname{co} \Omega$, namely in α. We so arrive at

THEOREM 3.1 (Runge). *Let Ω be an open subset of $\mathbb{R}^{m+1}$ and α be a subset of* $\operatorname{co} \Omega$ *having one point in each component of* $\operatorname{co} \Omega$. *Then $M_k(\mathbb{R}^{m+1}; \mathscr{A}) \oplus \mathscr{R}^*(\alpha)$ is dense in $M_k(\Omega; \mathscr{A})$ for the topology of uniform compact convergence. In particular, if* $\operatorname{co} \Omega$ *is connected and unbounded, $M_k(\mathbb{R}^{m+1}; \mathscr{A})$ is dense in $M_k(\Omega; \mathscr{A})$.*

Proof. Call for each $j = 1, 2, \ldots$,

$$K_j = \{x \in \Omega : | x | \leqslant j \text{ and } d(x, \operatorname{co} \Omega) \geqslant 1/j\}.$$

Then $(K_j)_{j=1}^{\infty}$ is a compact exhaustion of Ω. Decompose $\operatorname{co} K_j$ in its components $\Omega_0^j, \Omega_1^j, \Omega_2^j, \ldots$, with Ω_0^j the unbounded one. Since each component of $\operatorname{co} K_j$ contains a component of $\operatorname{co} \Omega$, $\Omega_i^j \cap \alpha \neq \varnothing$ for all $i = 1, 2, \ldots$. Take $a_i^j \in \Omega_i^j \cap \alpha$, $i = 1, 2, \ldots$, and call $\mathbf{a}_j = (a_i^j)$. Now let $f \in M_k(\Omega; \mathscr{A})$, p_{K_j} be the seminorm on $M_k(\Omega; \mathscr{A})$ associated to K_j and $\epsilon > 0$. Then clearly $f \in M_k(K_j; \mathscr{A})$ so that, in view of Lemma 3.3, there exists a function $h_j \in M_k(\mathbb{R}^{m+1}; \mathscr{A}) \oplus \mathscr{R}^*(\mathbf{a}_j)$, having its singularities in $\mathbf{a}_j \subset \alpha$, such that $p_{K_j}(f - h_j) = \sup_{x \in K_j} | f(x) - h_j(x)|_0 \leqslant \epsilon$. ∎

Remark. In fact we have that $\mathscr{P} \oplus \mathscr{R}^*(\alpha)$ is dense in $M_k(\Omega; \mathscr{A})$.

THEOREM 3.2 (Hartogs–Rosenthal) *Let K be a compact subset of $\mathbb{R}^{m+1}$ with Lebesgue measure zero and $\mathbf{a}$ be a subset of* $\operatorname{co} K$ *having one point in each bounded component of* $\operatorname{co} K$. *Then $M_k(\mathbb{R}^{m+1}; \mathscr{A}) \oplus \mathscr{R}^*(\mathbf{a})$ is uniformly dense in $C_0(K; \mathscr{A})$.*

Proof. Let μ be an $\mathscr{A}$-valued measure in $\mathbb{R}^{m+1}$ supported on K such that μ annihilates $M_k(\mathbb{R}^{m+1}; \mathscr{A}) \oplus \mathscr{R}^*(\mathbf{a})$. Then it follows from Lemma 3.3 that μ is zero on $M_k(K; \mathscr{A})$. Consequently $\mu * E_k = 0$ in $\operatorname{co} K$ so that, K having Lebesgue measure zero, $\mu * E_k = 0$ l.a.e. in $\mathbb{R}^{m+1}$. Theorem 4.3 in [6] then tells us that $\mu = 0$. ∎

Remark. If K has Lebesgue measure zero, we have in fact that $\mathscr{P} \oplus \mathscr{R}^*(\mathbf{a})$ is uniformly dense in $C_0(K; \mathscr{A})$.

4. Runge's Theorem for Regular Solutions at Infinity

If K is a compact subset of $\mathbb{R}^{m+1}$, then we have introduced in [6] the right $\mathscr{A}$-module $\tilde{M}_k(\text{co}\, K; \mathscr{A})$ consisting of all solutions of D^k in co K which are regular at infinity with respect to E_k. It has been shown that $\tilde{M}_k(\text{co}\, K; \mathscr{A})$ is a closed submodule of $M_k(\text{co}\, K; \mathscr{A})$ and that each $f \in \tilde{M}_k(\text{co}\, K; \mathscr{A})$ may be represented by a Cauchy type formula off some compact neighborhood K_1 of K.

The aim of this section is to establish a Runge-type theorem for the elements in $\tilde{M}_k(\text{co}\, K; \mathscr{A})$.

From Theorem 3.1 we already know that $M_k(\mathbb{R}^{m+1}; \mathscr{A}) \oplus \mathscr{R}^*(\alpha)$ is dense in $M_k(\Omega; \mathscr{A})$ and that, in the particular case where co Ω is connected and unbounded, $M_k(\mathbb{R}^{m+1}; \mathscr{A})$ is dense in $M_k(\Omega; \mathscr{A})$.

Now it will be proved that if co $\Omega = K$ is compact and hence bounded, then $\mathscr{R}^*(\alpha)$ is dense in $\tilde{M}_k(\text{co}\, K; \mathscr{A})$. For the case of holomorphic functions, these results may be interpreted as follows. Let us repeat that if $m = n = k = 1$, then $\mathscr{A} \simeq \mathbb{C}$, while D is nothing else but the Cauchy–Riemann operator $\bar{\partial}$. In this case $M_k(\mathbb{R}^{m+1}; \mathscr{A})$ replaces the space of entire functions. Moreover $\tilde{M}_k(\text{co}\, K; \mathscr{A})$ stands for the set of functions f which are holomorphic in co K and for which $\lim_{z \to \infty} f(z) = 0$ (see [2]).

We start with a lemma which is analogous to Lemma 3.1.

Lemma 4.1. *Let K be a compact subset of $\mathbb{R}^{m+1}$, ω be an open bounded neighborhood of K such that each component of ω intersects K. If μ is an $\mathscr{A}$-valued measure with compact support contained in* co ω, *then $\int d\mu(x) \times f(x) = 0$ for all $f \in \tilde{M}_k(\text{co}\, K; \mathscr{A})$ iff $\mu * E_k = 0$ in ω.*

Proof. As to the necessary condition, take $a \in K$ and consider the function $E_k(a - x) \in \tilde{M}_k(\text{co}\, K; \mathscr{A})$. Then

$$\begin{aligned}\mu * E_k(a) &= \int d\mu(x)\, E_k(a - x) \\ &= 0.\end{aligned}$$

Now let $\omega = \bigcup_{i \in \mathbb{N}} \omega_i$ be the decomposition of ω in its components and take for each $i \in \mathbb{N}$, $a_i \in \omega_i \cap K$. Then $\mu * E_k(a_i) = 0$. Consequently, as $\mu * E_k$ is an $\mathscr{A}$-valued analytic function in co$[\mu]$, $\mu * E_k = 0$ in each ω_i and so $\mu * E_k = 0$ in ω.

Conversely, first remark that in co$[\mu]$, $(\mu * E_k)\, D = \mu * E_k D = \mu * E_{k-1}$ so that from $\mu * E_k = 0$ in ω, it follows that $\mu * E_{k-1} = \mu * E_{k-2} = \cdots = \mu * E_1 = 0$ in ω.

Now let K_n be a suitable compact neighborhood of K which is contained in ω. Then in view of Cauchy's representation formula (see

[6, Corollary to Theorem 4.2]), for each $f \in \tilde{M}_k(\text{co}\, K; \mathscr{A})$ and $x \in K_\eta$,

$$f(x) = \int_{\partial K_\eta} \sum_{j=0}^{k-1} (-1)^{j+1} E_{j+1}(t-x)\, d\sigma_t\, D^j f(t).$$

Hence, using the fact that $\mu * E_{j+1} = 0$ in ω ($j = 0, 1, \ldots, k-1$), we obtain that

$$\begin{aligned}
\int d\mu(x) f(x) &= \int_{[\mu]} d\mu(x) \left[\int_{\partial K_\eta} \sum_{j=0}^{k-1} (-1)^{j+1} E_{j+1}(t-x)\, d\sigma_t\, D^j f(t) \right] \\
&= \int_{\partial K_\eta} \sum_{j=0}^{k-1} (-1)^{j+1} \left[\int d\mu(x)\, E_{j+1}(t-x) \right] d\sigma_t\, D^j f(t) \\
&= \int_{\partial K_\eta} \sum_{j=0}^{k-1} (-1)^{j+1} \mu * E_{j+1}(t)\, d\sigma_t\, D^j f(t) \\
&= 0.
\end{aligned}$$

This proves the sufficient condition since $f \in \tilde{M}_k(\text{co}\, K; \mathscr{A})$ has been taken arbitrarily. ∎

Using the notations of the previous section, we have

THEOREM 4.1. *Let K be a compact subset of $\mathbb{R}^{m+1}$ and α be a subset of K having one point in each component of K. Then $\mathscr{R}^*(\alpha)$ is dense in $\tilde{M}_k(\text{co}\, K; \mathscr{A})$ for the topology of uniform compact convergence.*

Proof. First we remark that $\mathscr{R}^*(\alpha)$ is a submodule of $M_k(\text{co}\, K; \mathscr{A})$ since each of its elements is a right $\mathscr{A}$-linear combination of functions having the form $\partial^{p-s} E_k(x - a_i)\, D^s / \partial x_{l_1} \cdots \partial x_{l_{p-s}}$, where $p \in \mathbb{N}$, $0 \leqslant s \leqslant \min(p, k-1)$, $(l_1, \ldots, l_{p-s}) \in \{1, \ldots, m\}^{p-s}$ and $a_i \in \alpha$. To prove that $\mathscr{R}^*(\alpha)$ is dense in $\tilde{M}_k(\text{co}\, K; \mathscr{A})$, by the Hahn–Banach and Riesz representation theorems, it clearly suffices to show that if T is a bounded right $\mathscr{A}$-linear functional on $C_0(K; \mathscr{A})$ which annihilates $\mathscr{R}^*(\alpha)$, then it vanishes on $\tilde{M}_k(\text{co}\, K; \mathscr{A})$ too.

To this end. let again $(K_j)_{j \in \mathbb{N}}$ be the compact exhaustion of $\Omega = \text{co}\, K$ considered so far and suppose that T is a right $\mathscr{A}$-linear functional on $C_0(K; \mathscr{A})$ bounded by some p_{K_j}, which annihilates $\mathscr{R}^*(\alpha)$. Furthermore, choose $j \in \mathbb{N}$ large enough such that K and $\{x: d(x, K) \leqslant 1/j\}$ both are contained in $\{x \in \mathbb{R}^{m+1}: |x| < j\}$. Then there exists an $\mathscr{A}$-valued measure μ supported on K_j such that

$$T(f) = \int d\mu(x) f(x), \qquad f \in C_0(\text{co}\, K; \mathscr{A}).$$

Now let co $K_j = \Omega_0{}^j \cup \Omega_1{}^j \cup \Omega_2{}^j \cup \cdots$ be the decomposition of co K_j in its components, $\Omega_0{}^j$ being the unbounded one. Then $K \subset \omega = \bigcup_{i\geqslant 1} \Omega_i{}^j$, ω being a bounded open neighborhood of K such that each component of ω contains at least one component of K.

Hence, for each $i = 1, 2,..., \Omega_i{}^j \cap \alpha \neq \varnothing$ so that, taking $a_i{}^j \in \Omega_i{}^j \cap \alpha$, $i = 1, 2,..., \mathbf{a}_j = (a_i{}^j)_{i\geqslant 1} \subset \alpha$.

As $\mu * E_k$ is an $\mathscr{A}$-valued analytic function in co$[\mu]$, for each $i = 1, 2,...$, there exists an open neighborhood $\Omega_{a_i{}^j}$ of $a_i{}^j$ in which $\mu * E_k$ admits a Taylor development. But, as by assumption,

$$\begin{aligned} \left.\frac{\partial^{p-s}(\mu * E_k)\, D^s}{\partial x_{l_1} \cdots \partial x_{l_{p-s}}}\right|_{x=a_i{}^j} &= \mu * \frac{\partial^{p-s} E_k D^s}{\partial x_{l_1} \cdots \partial x_{l_{p-s}}} (a_i{}^j) \\ &= \int d\mu(x) \frac{\partial^{p-s} E_k(a_i{}^j - x)\, D^s}{\partial x_{l_1} \cdots \partial x_{l_{p-s}}} \\ &= 0, \end{aligned}$$

we have that $\mu * E_k = 0$ in $\Omega_{a_i{}^j}$.

Consequently, $\Omega_i{}^j \subset$ co$[\mu]$ being connected, we obtain that $\mu * E_k = 0$ in $\Omega_i{}^j$ so that $\mu * E_k = 0$ in $\omega = \bigcup_{i\geqslant 1} \Omega_i{}^j$.

Taking account of Lemma 4.1 we find that $\int d\mu(x) f(x) = 0$ for all $f \in \tilde{M}_k(\text{co } K; \mathscr{A})$. ∎

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